The gauge-invariant Green function in 3+1 dimensional QED (QCD) and 2+1 dimensional Abelian (non-Abelian) Chern–Simon theory

2004 J. Phys.: Condens. Matter 16 4465

(http://iopscience.iop.org/0953-8984/16/25/007)

J. Phys.: Condens. Matter **16** (2004) 4465–4476 PII: S0953-8984(04)76911-8

The gauge-invariant Green function in 3 + 1 dimensional QED (QCD) and 2 + 1 dimensional Abelian (non-Abelian) Chern–Simon theory

Jinwu Ye

Department of Physics and Material Research Institute, The Pennsylvania State University, University Park, PA 16802, USA

Received 25 February 2004
Published 11 June 2004
Online at stacks.iop.org/JPhysCM/16/4465
doi:10.1088/0953-8984/16/25/007

Abstract
By applying the simple and effective method developed to study the gauge-invariant fermion Green function in 2 + 1 dimensional non-compact QED, we study the gauge-invariant Green function in 3 + 1 dimensional QED and 2 + 1 dimensional non-compact Chern–Simon theory. We also extend our results to the corresponding $SU(M)$ non-Abelian gauge theories. Implications for the fractional quantum Hall effect are briefly discussed.

1. Introduction

In any systems of gauge fields (Abelian or non-Abelian) coupled to matters (fermions or bosons), the conventional Green function is defined as

$$G(x, y) = \langle \psi(x)\bar{\psi}(y)\rangle. \tag{1}$$

In momentum space, the fully-interacting fermion Green function in equation (1) takes the form $G(k) = \mathrm{i}k_\mu\gamma_\mu/k^{2+\eta}$. In 4 (3) dimensional space–time, it corresponds to $\gamma_\mu x_\mu/x^{4-\eta}$ ($\gamma_\mu x_\mu/x^{3-\eta}$) where the anomalous dimension η can be calculated by the standard renormalization group (RG) by extracting UV divergences. Unfortunately, this Green function is not gauge invariant; η depends on the fixed gauge in which the calculation is done.

Schwinger proposed the following gauge-invariant Green function [1]:

$$G^{\mathrm{inv}}(x, y) = \langle \psi(x)\mathrm{e}^{\mathrm{i}e\int_x^y a_\mu(\xi)\,\mathrm{d}\xi_\mu}\bar{\psi}(y)\rangle \tag{2}$$

where the inserted Dirac string makes the Schwinger Green function gauge invariant. G^{inv} depends on the integral path $\mathcal{C}$ from x to y. For simplicity reasons, we take $\mathcal{C}$ to be a straight line[1].

In fact, the path $\mathcal{C}$ should be determined by the underlying physical systems. In QED or QCD, in principle, one should perform an average over all possible paths from x to y with

[1] For non-Abelian gauge theory, the inserted Dirac string should be path ordered.

some weights, but what kind of weight factors should be used is still unclear. Non-smooth paths with cusps or intersections will also cause additional complications. Perturbative QCD makes sense only at short distance, so choosing a straight line may be reasonable. See [2] for some preliminary discussions. In condensed matter systems such as fractional quantum Hall effects or high temperature superconductors (HTS), formulated on a lattice, the physical quantities such as the real electron Green function in FQHE or angle resolved photo-emissions spectroscopy (ARPES) in HTS are independent of path, therefore taking a straight line is not only the simplest but is also plausible. In [26], we will discuss the path dependence of gauge-invariant Green functions in different condensed matter systems formulated on lattices.

In quantum chromodynamics (QCD), the Schwinger gauge-invariant Green function is closely associated with the hadronic bound states, and it has been studied before [2]. In recent years, its importance in condensed matter system was recognized in the contexts of fractional quantum Hall systems [3] and high temperature superconductors [4–6].

By a singular gauge transformation which attaches two flux quanta to each electron, an electron in an external magnetic field was mapped to a composite fermion in a reduced magnetic field [7–9]. Although transport properties which are directly related to two particle Green functions can be directly studied in the composite fermion language, the tunnelling density of states which is directly related to the single particle electron Green function is much more difficult to study. In fact, the single particle electron Green function is equal to the gauge-invariant Green function of the composite fermion which was evaluated for non-relativistic fermions by phenomenological arguments in [3].

Most recently, the importance of the gauge-invariant Green function of a fermion to angle resolved photoemission (ARPES) data in high temperature superconductors was discovered independently by Rantner and Wen (RW) [4] and the author [5] in different contexts. Starting from the $U(1)$ or $SU(2)$ gauge theory of doped Mott insulators [10–12], RW discussed the relevance of this gauge-invariant Green function to ARPES data. They also pointed out that a in temporal gauge, the equal-space gauge-invariant Green function in equation (2) is equal to that of the conventional gauge-dependent one in equation (1). Starting from a complementary (or dual) approach pioneered by Balents *et al* [13], the author studied how quantum [6] or thermal [5] fluctuation generated $hc/2e$ vortices can destroy d-wave superconductivity and evolve the system into an underdoped regime at $T = 0$ or a pseudo-gap regime at finite T. In the vortex plasma regime around the *finite* temperature Kosterlitz–Thouless transition [14], the vortices can be treated by classical hydrodynamics. By Anderson singular gauge transformation, which attaches the flux from the classical vortex to the quasi-particles of d-wave superconductors [15–17][2], the quasi-particles (spinons) are found to move in a random *static* magnetic field generated by the classical vortex plasma [5]. The electron spectral function $G(\vec{x}, t) = \langle C_\alpha(0, 0) C_\alpha^\dagger(\vec{x}, t)\rangle$ is the product of the classical vortex correlation function and the *gauge-invariant* Green function of the spinon in the random magnetic field. Technically, this static gauge-invariant Green function is different from the original dynamic Schwinger gauge-invariant Green function. Conceptually, both are single particle gauge-invariant Green functions and are physically measurable quantities.

Recently, two different methods have been developed to calculate the gauge-invariant Green function equation (2). The author in [18] evaluated it in a path integral representation. In [19], by applying the methods developed to study clean [20] and disordered [21] FQH transitions and superconductors to insulator transitions [22], the author developed a very simple and effective method to study the gauge-invariant Green function. In the context of $2 + 1$ dimensional massless quantum electro-dynamics (QED3) [23, 11, 12], the author studied

[2] For the analogy to and the difference from the singular gauge transformation in FQHE, see [6].

the gauge-dependent Green function both in a temporal gauge and in a Coulomb gauge. In the temporal gauge, the infrared divergence was found to be in the middle of the contour integral along the real axis. It was regularized by deforming the contour into a complex plane by physical prescription; the anomalous dimension was found and argued to be the same as that of the gauge-invariant Green function. However, in the Coulomb gauge, the IR divergence was found to be at the two ends of the contour integral along the real axis, and therefore unregularizable; the anomalous dimension was even not defined. This IR divergence was shown to be cancelled in any physical gauge-invariant quantities such as the β function and correlation length exponent ν as observed in [20–22]. The author also studied the gauge-invariant Green function directly by Lorentz covariant calculation with different gauge-fixing parameters and found that the exponent is independent of the gauge-fixing parameters and is exactly the same as that found in the temporal gauge.

In this paper, we apply the simple and effective method developed in [19] to study two interesting physical systems: 3 + 1 dimensional QED and 2 + 1 dimensional Chern–Simon theory and their corresponding $SU(M)$ non-Abelian counterparts. The importance of the first system is obvious in high energy physics. The second system is closely related to the high temperature behaviours of 3 + 1 dimensional QED or QCD. It may also describe fractional quantum Hall (FQH) transitions [9, 24, 20, 21]. The gauge-invariant Green function equation (2) is a relativistic analogue of the tunnelling density of state in an FQH system studied by the phenomenological method in [3].

The paper is organized as follows. In the next section, we study QED4 in both the temporal gauge and Lorentz covariant gauge; we also extend our results to non-Abelian $SU(M)$ QCD. In section 3, we study QED3 with the Chern–Simon term also in both the temporal gauge and Lorentz covariant gauge; we also extend our results to non-Abelian $SU(M)$ Chern–Simon theory. Finally, we reach our conclusions by summarizing our results in three simple and intuitive rules.

2. 3 + 1 dimensional QED

The standard 3 + 1 dimensional massless quantum electro-dynamics (QED4) Lagrangian is

$$\mathcal{L} = \bar{\psi}_a \gamma_\mu (\partial_\mu - \mathrm{i} e a_\mu) \psi_a + \tfrac{1}{4}(f_{\mu\nu})^2 \tag{3}$$

where ψ_a is a four component spinor, $a = 1, \ldots, N$ are N species of Dirac fermion, γ_μ are 4×4 matrices satisfying the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$.

In contrast to the QED3 studied in [19], the coupling constant e is marginal (or dimensionless) at the 4 space–time dimension. Straightforward perturbation suffices.

With the gauge-fixing term $(1/(2\alpha))(\partial_\mu a_\mu)^2$, the gauge-field propagator is

$$D_{\mu\nu} = \frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right). \tag{4}$$

By extracting the UV divergence, we find the anomalous exponent for the gauge-dependent Green function in equation (1)

$$\eta = -\frac{\alpha e^2}{8\pi^2}. \tag{5}$$

Note that in the Landau gauge $\alpha = 0$, η vanishes! In the usual textbooks, detailed calculations were given in the Feymann gauge $\alpha = 1$ with $\eta = -e^2/(8\pi^2)$. Obviously, η is a gauge-dependent quantity and its physical meaning in any Lorentz covariant gauges is not evident. In the following section, we will calculate η in two Lorentz non-covariant gauges: the temporal gauge and Coulomb gauge.

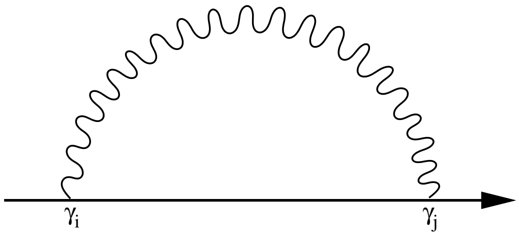

Figure 1. The fermion self-energy diagram in the temporal gauge.

2.1. The calculation in a temporal gauge

As stated in the introduction, the main focus of this paper is the gauge-invariant Green function equation (2). In a temporal gauge, the equal-space gauge-invariant Green function is the same as the gauge-dependent one [4].

The strategy taken in [19] is to calculate the conventional Green function in the temporal gauge $a_0 = 0$ and then see what we can say about the gauge-invariant Green function. As shown in [19] in the context of QED3, in the temporal gauge and Coulomb gauge which break Lorentz invariance, we run into both UV and IR divergences. A sensible and physical method was developed to regularize these plaguey IR divergences. Here we take the same strategy and apply a similar method to regularize these IR divergences in the context of QED4.

With the notation $K = (k_0, \vec{k})$, in the $a_0 = 0$ gauge, it is easy to find the propagator:

$$D_{ij}(K) = \frac{1}{K^2}\left(\delta_{ij} + \frac{k_i k_j}{k_0^2}\right). \tag{6}$$

The one-loop fermion self-energy Feymann diagram is shown in figure 1. The corresponding expression is

$$\Sigma(K) = -\mathrm{i}e^2 \int \frac{\mathrm{d}^3 Q}{(2\pi)^3}\gamma_i \frac{\gamma_\mu (K-Q)_\mu}{(K-Q)^2}\gamma_j \frac{1}{Q^2}\left(\delta_{ij} + \frac{q_i q_j}{q_0^2}\right). \tag{7}$$

By using standard γ matrices Clifford algebra and suppressing the prefactor $-\mathrm{i}e^2$, we can simplify the above equation to

$$\Sigma(K) = \gamma_0 \int \frac{\mathrm{d}^4 Q}{(2\pi)^4}\frac{(k-q)_0}{(K-Q)^2 Q^2}\left(3 + \frac{\vec{q}^{\,2}}{q_0^2}\right) + \gamma_i \int \frac{\mathrm{d}^4 Q}{(2\pi)^4}\frac{(k+q)_i}{(K-Q)^2 Q^2} + \gamma_i \int \frac{\mathrm{d}^4 Q}{(2\pi)^4}\frac{\vec{q}^{\,2}(k+q)_i - 2\vec{q}\cdot\vec{k} q_i}{(K-Q)^2 Q^2 q_0^2}. \tag{8}$$

We choose the external momentum K to be along the z axis, then $Q_4 = Q\cos\theta$, $Q_3 = Q\cos\theta_1\sin\theta$, $Q_2 = Q\cos\phi\sin\theta_1\sin\theta$, $Q_1 = Q\sin\phi\sin\theta_1\sin\theta$, $\mathrm{d}^4 Q = Q^2\,\mathrm{d}Q\,\mathrm{d}\phi\sin\theta_1\,\mathrm{d}\theta_1\sin^2\theta\,\mathrm{d}\theta$. Setting $x = -\cos\theta$, we find the logarithmic divergence:

$$\frac{\gamma_0 k_0}{4\pi^3}\int_{-1}^{1}\mathrm{d}x\,\sqrt{1-x^2}(-4x^2 + x^{-2})\log\Lambda. \tag{9}$$

The integral can be rewritten as

$$\int_{-1}^{1}\mathrm{d}x\,\frac{\sqrt{1-x^2}}{x^2} - \frac{\pi}{2}. \tag{10}$$

As expected, we run into IR divergence at $x = 0$, which is in the middle point of the contour integral on the real axis from -1 to 1. Fortunately, by physical prescription, we can avoid the IR singularity at $x = 0$ by deforming the contour as shown in figure 2.

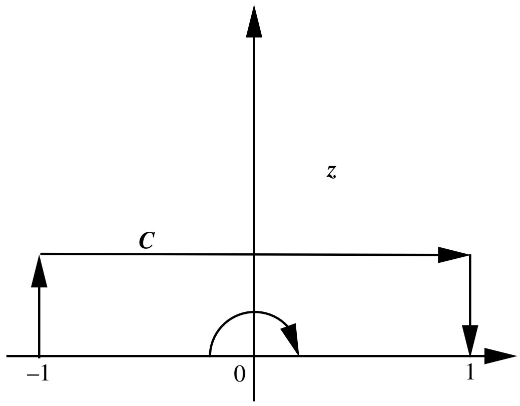

Figure 2. The contour path $\mathcal{C}$ to bypass the IR singularity in the temporal gauge.

The divergent part in equation (10) becomes

$$\int_{-1}^{1} \mathrm{d}z\, \frac{\sqrt{1-x^2}}{x^2} = \int_{\mathcal{C}} \mathrm{d}z\, \frac{\sqrt{1-z^2}}{z^2} = -\pi. \tag{11}$$

Putting back the prefactor $-\mathrm{i}e^2$, we get the final answer:

$$-\mathrm{i}e^2 \frac{\gamma_0 k_0}{4\pi^3}\left(-\frac{3\pi}{2}\right)\log\Lambda = \mathrm{i}\gamma_0 k_0 \frac{3e^2}{8\pi^2}\log\Lambda. \tag{12}$$

We can identify the anomalous dimension as

$$\eta = \frac{3e^2}{8\pi^2}. \tag{13}$$

We expect that this is the correct anomalous dimension of the gauge-invariant Green function equation (2) in QED4. It is consistent with the result in [2, 18] achieved with a very different method.

In the above calculation, we choose a cut-off Λ in 4-momentum Q. Just as in QED3 studied in [19], we can introduce an alternate cut-off $\tilde{\Lambda}$ only in 3-momentum $\vec{q}$, but integrate the frequency q_0 freely from $-\infty$ to ∞. Using the similar IR regularization scheme as in figure 2, we find exactly the same answer as equation (13). This agreement indeed shows that the exponent equation (13) is universal, and independent of different cut-offs or different renormalization schemes.

Just as in QED3 studied in [19], we could evaluate the Green function in the Coulomb gauge $\partial_i a_i = 0$. As in the temporal gauge, we also run into IR divergence. In both cut-off $Q < \Lambda$ and cut-off $q < \tilde{\Lambda}$, we run into IR divergences at $x = \pm 1$, which are at the two end points of the contour integral on the real axis from -1 to 1. Unfortunately, from physical prescription, we are unable to avoid the IR singularities at the two end points $x = \pm 1$ by deforming the contour. Therefore, we are unable to identify the anomalous exponent. Furthermore, the results are different in the two different cut-offs. This should cause no disturbance, because, in contrast to the Green function in the temporal gauge, the Green function equation (1) in the Coulomb gauge does not correspond to any physical quantities. All these IR divergences should disappear in any gauge-invariant physical quantities like the β function and critical exponents. The final answers for these physical quantities should also be independent of different cut-offs or different renormalization schemes.

2.2. Lorentz covariant calculation

In this subsection, we will calculate the gauge-invariant Green function directly in the Lorentz covariant gauge equation (4) without resorting to the gauge-dependent Green function. We will also compare with the result achieved in the temporal gauge.

The inserted Dirac string in equation (2) can be written as

$$\int_x^y a_\mu(\xi)\,\mathrm{d}\xi_\mu = \int a_\mu(x) j^s_\mu(x)\,\mathrm{d}^d x \tag{14}$$

where the source current is

$$j^s_\mu(x) = \int_{\mathcal{C}} \mathrm{d}\tau\, \frac{\mathrm{d}\xi_\mu}{\mathrm{d}\tau}\delta(x_\mu - \xi_\mu(\tau)) \tag{15}$$

where τ parameterizes the integral path $\mathcal{C}$ from x to y.

We will follow the procedures outlined in detail in [19]. (1) Combining the source current with the fermion current $j_\mu(x) = \bar\psi(x)\gamma_\mu\psi(x)$ to form the total current $j^t_\mu(x) = j_\mu(x) + j^s_\mu(x)$ (2) Integrating out a_μ in the Lorentz covariant gauge equation (4), we find

$$G^{\mathrm{inv}}(x,y) = \frac{1}{Z}\int \mathcal{D}\psi\mathcal{D}\bar\psi\,\psi(x)\bar\psi(y)\mathrm{e}^{-\int \mathrm{d}^d x\,\bar\psi\gamma_\mu\partial_\mu\psi}\mathrm{e}^{-W} \tag{16}$$

where Z is the partition function of QED4 and W is given by

$$\begin{aligned} W = \frac{e^2}{2}\int \mathrm{d}x\,\mathrm{d}x'(&j_\mu(x)D_{\mu\nu}(x-x')j_\nu(x') \\ &+ j^s_\mu(x)D_{\mu\nu}(x-x')j^s_\nu(x') \\ &+ 2j_\mu(x)D_{\mu\nu}(x-x')j^s_\nu(x')). \end{aligned} \tag{17}$$

The first term in equation (17) is just the conventional long-range four-fermion interaction mediated by the gauge field; it leads to the anomalous exponent in the covariant gauge given in equation (5):

$$\eta_1 = -\frac{\alpha e^2}{8\pi^2}. \tag{18}$$

Note that in the Landau gauge $\alpha = 0$, η_1 simply vanishes!

The second term in equation (17) is given by

$$-\frac{e^2}{(2\pi)^4}\int \frac{\mathrm{d}^4k}{k^4}\frac{k^2(y-x)^2 - (1-\alpha)(k(y-x))^2}{(k(y-x))^2}(1-\cos k(y-x)). \tag{19}$$

Extracting the leading terms as $\Lambda \to \infty$ turns out to be a little bit more difficult than its 2 + 1 dimensional counterpart discussed in [19]. Fortunately, the α independent part was already given in [25] in a different context:

$$-\frac{e^2}{4\pi^2}\Lambda r + \frac{3e^2}{8\pi^2}\log\Lambda r + \frac{3e^2}{8\pi^2}\left(\gamma - \log 2 + \frac{1}{2}\right) \tag{20}$$

where γ is the Euler constant. The linear divergence is a artefact of the momentum cut-off and should be ignored in Lorentz invariant regularization.

We only need to evaluate the α dependent part:

$$-\frac{e^2}{4\pi^3}\alpha\int_0^\Lambda \frac{\mathrm{d}k}{k}\int_0^\pi \mathrm{d}\theta\,\sin^2\theta(1-\cos(kr\cos\theta)) = -\frac{e^2\alpha}{8\pi^2}\left(\log\Lambda r + \gamma - \frac{1+\log 4}{2}\right). \tag{21}$$

Combining the Logarithmic terms in equations (20) and (21) leads to

$$\eta_2 = \frac{3e^2}{8\pi^2} - \frac{e^2\alpha}{8\pi^2}. \tag{22}$$

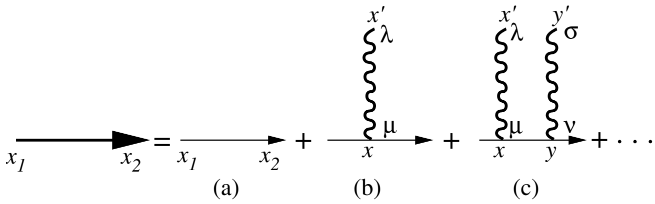

Figure 3. The perturbative expansion series of equation (23).

Finally, we should evaluate the contribution from the third term in equation (17), which can be written as

$$-e^2 \int \mathrm{d}x\, \bar{\psi}(x)\gamma_\mu \psi(x) \int_x^y \mathrm{d}x'_\nu\, D_{\mu\nu}(x-x'). \tag{23}$$

Equation (23) is essentially quadratic in the fermions. Combining it with the free fermion action leads to

$$\mathcal{S} = \int \mathrm{d}x\, \bar{\psi}(x)\left(\gamma_\mu\partial_\mu + e^2\gamma_\mu \int_{x_1}^{x_2} \mathrm{d}x'_\nu\, D_{\mu\nu}(x-x')\right)\psi(x). \tag{24}$$

In principle, the propagator of the fermion $\langle\psi(x_1)\bar{\psi}(x_2)\rangle$ can be calculated by inverting the quadratic form in the above equation, but this is not easy to carry out in practice. Instead we can construct the perturbative expansion in *real space* by the following Feymann diagrams in figure 3. The corresponding expression is

$$\begin{aligned} G(x_1,x_2) &= G_0(x_1,x_2) \\ &\quad - e^2 \int \mathrm{d}x\, G_0(x_1,x) \int_{x_1}^{x_2} \mathrm{d}x'_\lambda\, \gamma_\mu D_{\mu\lambda}(x-x') G_0(x,x_2) \\ &\quad + e^4 \int \mathrm{d}x\, G_0(x_1,x) \int_{x_1}^{x_2} \mathrm{d}x'_\lambda\, \gamma_\mu D_{\mu\lambda}(x-x') \\ &\quad \times \int \mathrm{d}y\, G_0(x,y) \int_{x_1}^{x_2} \mathrm{d}y'_\sigma\, \gamma_\nu D_{\nu\sigma}(y-y') G_0(y,x_2) + \cdots. \end{aligned} \tag{25}$$

Being just quadratic, there are no loops in the above Feymann series. But there may still be potential divergences.

The explicit expression for figure 3(b) is

$$F(x) = -e^2 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}q_1 x} - \mathrm{e}^{-\mathrm{i}q_2 x}}{-\mathrm{i}(q_1-q_2)x} G_0(q_1)\gamma_\mu x_\nu D_{\mu\nu}(q_1-q_2) G_0(q_2) \tag{26}$$

where $x_2 - x_1 = x$.

After some lengthy but straightforward manipulations, we can write equation (26) as the sum of two parts $F(x) = F_1(x) + F_2(x)$ with

$$\begin{aligned} F_1(x) = \mathrm{i}2e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \bigg(&\frac{\gamma_\mu k_\mu}{q^2(q-k)^2} + \frac{-2kx\gamma_\mu k_\mu + k^2\gamma_\mu x_\mu}{qxq^2(q-k)^2} \\ &+ \frac{kx\gamma_\mu q_\mu - kq\gamma_\mu x_\mu}{qxq^2(q-k)^2} \bigg) \end{aligned} \tag{27}$$

and

$$F_2(x) = -\mathrm{i}2e^2(1-\alpha) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{k^2\gamma_\mu q_\mu + q^2\gamma_\mu k_\mu}{(k+q)^4 q^2}. \tag{28}$$

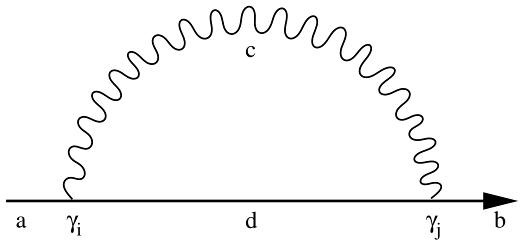

Figure 4. The fermion self-energy diagram in $SU(M)$ QCD in the temporal gauge.

It is easy to see that the second term in equation (27) is convergent, while the UV divergences in the third term in equation (27) exactly cancel as follows:

$$-\frac{xk\gamma_\mu x_\mu}{8\pi^2 x^2}\log\Lambda + \frac{xk\gamma_\mu x_\mu}{8\pi^2 x^2}\log\Lambda = 0. \tag{29}$$

The logarithmic divergence in the first term in equation (27) cancels exactly that of the α independent term in equation (28). Only the α dependent divergence in equation (28) survives:

$$\frac{\alpha e^2}{4\pi^2}\frac{\mathrm{i}\gamma_\mu k_\mu}{k^2}\log\Lambda. \tag{30}$$

The above equation leads to

$$\eta_3 = \frac{\alpha e^2}{4\pi^2}. \tag{31}$$

In all, the final anomalous exponent is

$$\eta = \eta_1 + \eta_2 + \eta_3 = \frac{3e^2}{8\pi^2}. \tag{32}$$

This is exactly the same as that calculated in the temporal gauge. As expected, the gauge-fixing parameter α drops out in the anomalous dimension η although η_1, η_2, η_3 all depend on α separately.

2.3. 3 + 1 dimensional non-Abelian $SU(M)$ QCD

The calculations in the last two subsections on Abelian QED can be straightforwardly extended to non-Abelian $SU(M)$ QCD by paying special attention to the internal group structure of the $SU(M)$ group.

In the temporal gauge, the one-loop fermion self-energy Feymann diagram in $SU(M)$ QCD is shown in figure 4. The corresponding expression is

$$\Sigma(K) = -\mathrm{i}e^2\int\frac{\mathrm{d}^3Q}{(2\pi)^3}\gamma_i\frac{\gamma_\mu(K-Q)_\mu}{(K-Q)^2}\gamma_j\frac{1}{Q^2}\left(\delta_{ij}+\frac{q_iq_j}{q_0^2}\right)(T^c)_{ab}(T^c)_{db} \tag{33}$$

where T^c with $c = 1, \ldots, M^2-1$ are M^2-1 generators of the $SU(M)$ group. $a, d, b = 1, \ldots, M$ are M colour indices of fermions transforming as a fundamental representation of the $SU(M)$ group.

It is easy to see

$$\Sigma^{ab}(\mathrm{QCD}) = (T^c)^2_{ab}\Sigma(\mathrm{QED}) \tag{34}$$

where $(T^c)^2_{ab} = C_2(F)\delta_{ab}$ with the quadratic Casimir $C_2(F) = \frac{M^2-1}{2M}$ for the fundamental representation of the $SU(M)$ group.

From equation (13), we have

$$\eta_{SU(M)} = \frac{M^2-1}{2M}\frac{3e^2}{8\pi^2}. \tag{35}$$

For $SU(3)$ QCD, $C_2(F) = 4/3$, then $\eta_{SU(3)} = e^2/(2\pi^2)$, which is consistent with the result achieved previously with a different method [2, 18].

Equation (35) can also be derived by using the Lorentz covariant calculation presented in the last subsection.

The results achieved in this section are not new, but two new, simple and effective methods are developed to rederive these old results. In the next section, we will use these new methods to derive new results in gauge theories with the Chern–Simon term.

3. 2 + 1 dimensional non-compact QED with the Chern–Simon term

In the two component notations suitable for describing the time reversal and parity breaking mass term and the Chern–Simon term, the standard 2 + 1 dimensional massless quantum electro-dynamics Lagrangian with the Chern–Simon term is

$$\mathcal{L} = \bar{\psi}_a\gamma_\mu\left(\partial_\mu - \mathrm{i}\frac{g}{\sqrt{N}}a_\mu\right)\psi_a + \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{36}$$

where $a = 1, \ldots, N$ are N species of Dirac fermion[3].

This Lagrangian was used to describe FQH transitions in [9, 24, 20, 21]. As shown in [24], in a straightforward perturbation expansion, there are no extra UV divergences from the CS term in one loop. One has to go to two loops to see the extra UV divergences from the CS term. Instead of going to two-loop calculations, we resort large N expansion by scaling the coupling constant as $g/\sqrt{N}$ in equation (36). Integrating out N pieces of fermions generates an additional dynamic quadratic term for the gauge field:

$$\mathcal{S}_2 = \frac{1}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}a_\mu(-k)\left(\Pi^e(k)k\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Pi^o(k)\epsilon_{\mu\nu\lambda}k_\lambda\right)a_\nu(k). \tag{37}$$

To one-loop order [24]

$$\Pi^e = \frac{g^2}{16}, \qquad \Pi^o = 0. \tag{38}$$

The dynamics of the gauge field is given by

$$\mathcal{L} = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{1}{2}a_\mu(-k)\Pi^e(k)k\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)a_\nu(k). \tag{39}$$

Adding the gauge-fixing term $(1/(2\alpha))(\partial_\mu a_\mu)^2$, we can get the propagator of the gauge field in the covariant gauge:

$$D_{\mu\nu}(k) = -A\frac{\epsilon_{\mu\nu\lambda}k_\lambda}{k^2} + \frac{B}{k}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{\alpha k_\mu k_\nu}{k^4} \tag{40}$$

where $A = 1/(1+\Pi_e^2(k))$, $B = \Pi_e(k)/(1+\Pi_e^2(k))$.

Changing the last local gauge-fixing term to a non-local gauge fixing term [23] leads to

$$D_{\mu\nu}(k) = -A\frac{\epsilon_{\mu\nu\lambda}k_\lambda}{k^2} + \frac{B}{k}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right). \tag{41}$$

In the following, we follow the same procedures in [19], paying special attention to the extra effects due to the CS term.

[3] In this paper, I only consider non-compact 2+1 dimensional CS gauge theory and leave the compact case for future work [26].

3.1. The calculation in the temporal gauge

With the notation $K = (k_0, \vec{k})$, in the $a_0 = 0$ gauge, we can invert equation (39) to find the propagator:

$$D_{ij}(K) = -A\frac{\epsilon_{ij}}{k_0} + \frac{B}{K}\left(\delta_{ij} + \frac{k_i k_j}{k_0^2}\right). \tag{42}$$

It is known that the antisymmetric CS propagator does not contribute to divergence to one-loop order [24]; the result in [19] can be directly applied by replacing $16/N$ by $(Bg^2)/N$:

$$\eta = \frac{4Bg^2}{3\pi^2 N}. \tag{43}$$

We expect that this is the correct anomalous dimension of the gauge-invariant Green function equation (2) in 2 + 1 dimensional Chern–Simon theory.

The tunnelling density of state $\rho(\omega) \sim \omega^{1-\eta}$. That η is positive indicates the tunnelling DOS *increases* due to the Chern–Simon interaction.

3.2. Lorentz covariant calculation

In this section, we will calculate the gauge-invariant Green function directly in the Lorentz covariant gauge equation (41) without resorting to the gauge-dependent Green function. We will also compare with the result achieved in the temporal gauge.

First, let us see what is the contribution from the first term in equation (17). As shown in [24], the antisymmetric CS propagator in equation (41) does not lead to divergence; the symmetric part of the propagator leads to

$$\eta_1 = \frac{g^2 B}{N}\frac{1}{3\pi^2}(1 - 3\alpha/2). \tag{44}$$

In evaluating the contribution from the second term in equation (17), the extra piece due to the CS term is

$$\frac{Ag^2}{(2\pi)^3 N}\int \frac{\mathrm{d}^3 k}{k^3}\frac{\epsilon_{\mu\nu\lambda}k_\lambda}{k^2}(y-x)_\mu (y-x)_\nu (1 - \cos k(y-x)). \tag{45}$$

Obviously, this extra term vanishes due to the antisymmetric ϵ tensor. The result in [19] can be directly applied:

$$\eta_2 = \frac{Bg^2}{N}\frac{1}{\pi^2}(1 - \alpha/2). \tag{46}$$

Finally, in evaluating the contribution from figure 3(b), the extra term due to the CS term is

$$F(x)_{cs} = \frac{Ag^2}{N}\int \frac{\mathrm{d}^3 q_1}{(2\pi)^3}\frac{\mathrm{d}^3 q_2}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}q_1 x} - \mathrm{e}^{-\mathrm{i}q_2 x}}{-\mathrm{i}(q_1 - q_2)x} G_0(q_1)\frac{\gamma_\mu x_\nu \epsilon_{\mu\nu\lambda}(q_1 - q_2)_\lambda}{(q_1 - q_2)^2} G_0(q_2) \tag{47}$$

where $x_2 - x_1 = x$.

After straightforward manipulation, the above equation can be simplified to

$$-2\mathrm{i}\frac{g^2 A}{N}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}kx}}{k^2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{kxq^2 - kqqx}{qxq^2(q-k)^2}. \tag{48}$$

It is easy to find that the potential UV divergences in the first term and second term, in fact, vanish separately!

Again, the result in [19] can be directly applied:

$$\eta_3 = \frac{Bg^2}{N}\frac{\alpha}{\pi^2}. \tag{49}$$

In all, the final anomalous exponent is

$$\eta = \eta_1 + \eta_2 + \eta_3 = \frac{4Bg^2}{3\pi^2 N}. \tag{50}$$

This is exactly the same as that calculated in the temporal gauge.

Although evaluating all these $(1/N)^2$ corrections is beyond the scope of this paper, we have established firmly our results (equation (50)) to order $1/N$.

3.3. Non-Abelian $SU(M)$ Chern–Simon gauge theory

It is useful to extend our results on Abelian CS theory established in the two previous subsections to non-Abelian CS theory. Non-Abelian gauge theories arise both in high temperature superconductors [10] and FQHE [28]. In contrast to the state at filling factor $\nu = 1/2$ which has a gapless Fermi surface, the state at filling factor $5/2$ may be a gapped paired quantum Hall state [28]. This paired state may be a non-Abelian state where the quasi-particles may obey non-Abelian statistics [28]. The effective low energy theory of non-Abelian states can be described by $SU(2)$ Chern–Simon gauge theory.

Extension to relativistic $SU(M)$ non-Abelian CS theory discussed in [27] is straightforward. By using the rule derived in section 2.3, to the order of $1/N$, we get the result

$$\eta = \frac{M^2 - 1}{2M} \frac{g^4}{1 + (\frac{g^2}{16})^2} \frac{1}{12\pi^2 N}. \tag{51}$$

This result could be achieved both in the temporal gauge and in the Lorentz covariant gauge presented in the previous two subsections.

4. Conclusions

In this paper, using the simple and powerful methods developed in [19], we calculated the gauge-invariant fermion Green function in $3+1$ dimensional QED and $2+1$ dimensional QED with the Chern–Simon term, and their corresponding $SU(M)$ non-Abelian counterparts by different methods. The calculations in the temporal gauge with different cut-offs and Lorentz covariant calculation with different gauge-fixing parameters α lead to the same answers. These facts strongly suggest that equation (13) is the correct exponent to one loop and equation (43) is the correct exponent in the leading order of $1/N$. These methods was previously applied to study 2+1 dimensional QED and have also been applied to many different physical systems [26]. We summarize our results with the following three useful rules of thumb.

Rule 1: *In the temporal gauge, IR divergence is always in the middle of the integral and can be regularized by deforming the contour. The finite exponent is the anomalous exponent of the gauge-invariant Green function.*

Rule 2: *In the Coulomb gauge, IR divergence is always at the two ends of the contour integral and cannot be regularized by deforming the contour. The anomalous exponent is not even defined, but the IR divergence will be cancelled in any gauge-invariant physical quantity.*

Rule 3: *The anomalous exponent of the gauge-invariant Green function is equal to the sum of the exponent of the gauge-dependent Green function in the* Landau gauge *and the exponent of the inserted Dirac string also in the* Landau gauge.

Although we have demonstrated the above three rules of thumb only to one loop or to the order $1/N$, we expect that they all hold to any loops or to any order of $1/N$.

Acknowledgment

I thank J K Jain for helpful discussions.

References

[1] Schwinger J 1959 *Phys. Rev. Lett.* **3** 296
[2] Craigie N S and Dorn H 1981 *Nucl. Phys.* B **185** 204
Stefanis N G 1984 *Nuovo Cimento* A **83** 205 Very recently, these two references were brought to the author's attention from the references cited in [20]. The methods employed in these two references are different than those developed by the author in [21]
[3] He S, Platzman P M and Halperin B I 1993 *Phys. Rev. Lett.* **71** 777
Shytov A V, Levitov L S and Halperin B I 1998 *Phys. Rev. Lett.* **80** 141 (*Preprint* cond-mat/0005016)
[4] Rantner W and Wen X G 2001 *Phys. Rev. Lett.* **86** 3871 (*Preprint* cond-mat/0105540)
[5] Ye J 2001 *Phys. Rev. Lett.* **87** 227003
[6] Ye J 2002 *Phys. Rev.* B **65** 214505
[7] Jain J K 1989 *Phys. Rev. Lett.* **63** 199
[8] Zhang S C, Hansson T and Kivelson S 1989 *Phys. Rev. Lett.* **62** 82
[9] Halperin B I, Lee P A and Read N 1993 *Phys. Rev.* B **47** 7312
[10] Ioffe L B and Larkin A L 1989 *Phys. Rev.* B **39** 8988
Nagaosa N and Lee P A 1990 *Phys. Rev. Lett.* **64** 2450
Nagaosa N and Lee P A 1992 *Phys. Rev.* B **45** 966
Ye J and Sachdev S 1991 *Phys. Rev.* B **44** 10173
Wen X G and Lee P A 1998 *Phys. Rev. Lett.* **80** 2193
Lee P A *et al* 1998 *Phys. Rev.* B **57** 6003
[11] Affleck I and Marston J B 1988 *Phys. Rev.* B **37** 3774
[12] Marston B 1990 *Phys. Rev. Lett.* **64** 1166
[13] Balents L, Fisher M P A and Nayak C 1998 *Int. J. Mod. Phys.* B **12** 1033
Balents L, Fisher M P A and Nayak C 1999 *Phys. Rev.* B **60** 1654
[14] Emery V J and Kivelson S A 1995 *Nature* **374** 434
Williams G 1999 *Phys. Rev. Lett.* **82** 1201
[15] Anderson P W 1998 *Preprint* cond-mat/9812063
[16] Ye J 2001 *Phys. Rev. Lett.* **86** 316
[17] Melnikov A S 2001 *Phys. Rev. Lett.* **86** 4198
[18] Khveshchenko D 2001 *Preprint* cond-mat/0112202
[19] Ye J 2003 *Phys. Rev.* B **67** 115104
[20] Ye J and Sachdev S 1998 *Phys. Rev. Lett.* **80** 5409
[21] Ye J 1999 *Phys. Rev.* B **60** 8290
[22] Ye J 1998 *Phys. Rev.* B **58** 9450
[23] Appelquist T *et al* 1988 *Phys. Rev. Lett.* **60** 2575
Nash D 1989 *Phys. Rev. Lett.* **62** 3024
[24] Chen W, Fisher M P A and Wu Y S 1993 *Phys. Rev.* B **48** 13749
[25] Popov V N 1983 *Functional Integrals in Quantum Field Theory and Statistical Physics* (Dordrecht: Reidel)
[26] Ye J, unpublished
[27] Chen W, Semenoff G W and Wu Y S 1992 *Phys. Rev.* D **46** 5521
[28] Moore G and Read N 1991 *Nucl. Phys.* B **360** 362
Milovanovic M and Read N 1996 *Phys. Rev.* B **53** 13559
Read N and Green D 2000 *Phys. Rev.* B **61** 10267